Summer Internship Project
Report

Nonlinear Dynamics

Submitted by

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Abstract

In this report, we will discuss the basics of nonlinear dynamics and will dive into the study of a nonlinear system - a population growth model involving different population interactions. Nonlinear dynamics involves the study of systems that are governed by equations that are not of the linear form - such as population dynamics, double pendulum, weather dynamics and fluid dynamics. We will begin our study with an analysis of first-order and linear systems, introducing the notion of fixed points and their stability and bifurcations and its different types. We will look into a few numerical methods to solve these systems and phase plane analysis. We will conclude by analyzing a second-order nonlinear system of two populations interacting with each other.

Contents

1 Introduction

1.1 Dynamical Systems

In this section, we will look into different dynamical systems, their mathematical forms and a few terminologies related to them. Two major types of dynamical systems are:-

- Differential equations - describes the evolution of systems in continuous time.
- Iterated maps - describes the evolution of systems in discrete time.

Differential equations can be further classified into ordinary and partial differential equations. Let us restrict our study to ordinary differential equations and non-linear phenomena arising in their domain.

1.2 Ordinary Differential Equations

A general framework for an n-dimensional or nth-order system of autonomous ordinary differential equations would be -

$$\dot{x_1} = f_1(x_1, ..., x_n)$$

$$\vdots$$

$$\dot{x_n} = f_n(x_1, ..., x_n)$$

The system is said to be non-autonomous if any of the functions - f_1,....,f_n - has a time dependence.
In such a case, we define:-

$$x_{n+1} = t$$

$$\implies \dot{x_{n+1}} = 1$$

This definition makes the system (n+1)-dimensional.

1.3 The Nonlinear World

A system is said to be nonlinear if any of the functions f_1,....,f_n has non-linear terms.
Such non-linear systems are difficult to deal with, as most of them are unsolvable analytically. Hence we rely on different numerical and graphical methods to obtain a solution, at least qualitatively.
We observe that the systems become more and more complex with an increase in the number of variables, ranging from the study of logistic population growth model and pendulum to neural networks and quantum field theory.

2 First Order Systems

In this section, we analyze the simplest nonlinear systems,i.e, the first order systems. Inspirations drawn from this study will help us tackle higher order systems. The general form of a one- dimensional or first-order autonomous system is -

$$\dot{x} = f(x) \tag{1}$$

2.1 Geometric Approach - Flow on a Line

We assume that an imaginary particle - **phase point** - is moving along the real line such that $\dot{x}$ gives the velocity of the particle at position x , the velocity is towards the increasing x-direction if $\dot{x} > 0$ and in the opposite direction if $\dot{x} < 0$. Hence the differential equation $\dot{x} = f(x)$ represents a vector field on the line.
Suppose a phase point starts from x_0 and is carried by the flow, then its position function $x(t)$ is called the **trajectory** based at x_0.
Phase portrait is the graphical representation of all qualitatively different trajectories of the system.

2.2 Fixed points and their Stability

x^* is called a fixed point of the system $\dot{x} = f(x)$ if $f(x^*) = 0$. Hence, there is no flow at fixed points.
So, they represent the equilibrium solutions of the differential equation since $x(0) = x^* \implies x(t) = x^*$ $\forall t \in [0, \infty)$.
Based on their stability, fixed points can be classified into three -

- **Stable fixed points** : Flow is towards these points.
- **Unstable fixed points** : Flow is away from these points.
- **Half-stable fixed points** : Flow continues through the point without changing direction.

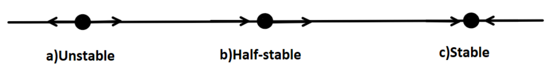

Figure 1: Different types of fixed points

2.3 Linear Stability Analysis

In this section we analyse the stability of fixed points by linearizing about them.

Let x^* be a fixed point and $\eta(t)$ denotes a small perturbation away from x^*.

$$\begin{aligned} \eta(t) &= x(t) - x^* \\ \implies \dot{\eta} &= \dot{x} = f(x) \\ \implies \dot{\eta} &= f(x^* + \eta) \end{aligned}$$

Using Taylor series expansion around x^*, we obtain -

$$\dot{\eta} = f(x^*) + \eta f'(x^*) + O(\eta^2)$$

Since $\eta(t)$ is a small perturbation, $O(\eta^2)$ can be neglected.(*Here, we assume that* $f'(x^*) \neq 0$. *If not,* $O(\eta^2)$ *can't be neglected.*) Also, $f(x^*) = 0$ since x^* is a fixed point. Hence,

$$\dot{\eta} \approx \eta f'(x^*) \tag{2}$$

Hence, the perturbation $\eta(t)$ -

- grows exponentially if $f'(x^*) > 0$, making x^* unstable.
- decays if $f'(x^*) < 0$, making x^* stable.

3 Bifurcations

3.1 Definition and some terminologies

Consider the first-order system $\dot{x} = f(x, r)$, where r is some parameter. If the qualitative nature of solutions when $r < r_0$ is different from that of the solutions when $r > r_0$, then the system is said to have undergone a **bifurcation** at $r = r_0$ and r_0 is called the **bifurcation point**. This qualitative change can arise due to the creation, destruction or change in stability of the fixed points.
At fixed points, $\dot{x} = 0 \implies f(x, r) = 0$. The curve $f(x, r) = 0$ drawn in the x-r plane is called the **bifurcation diagram** of the system.
For a particular type of bifurcation, the dynamics of any system undergoing that bifurcation will be similar to certain prototypical forms, in a neighbourhood of the fixed point. Such forms are called the **normal forms** of that bifurcation.
Let us look into certain types of bifurcations.

3.2 Saddle-Node Bifurcation

In saddle-node bifurcation, as a parameter is varied, two fixed points move towards each other, collide and mutually annihilate.
The normal forms of saddle-node bifurcation are - $\dot{x} = r - x^2$ or $\dot{x} = r + x^2$. Both the systems undergo a saddle-node bifurcation at $r = 0$(*as shown in fig(2)*).

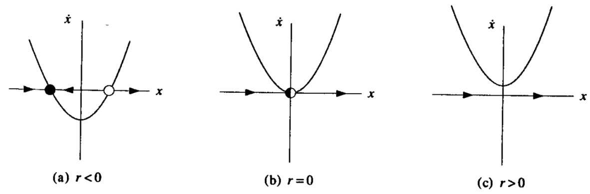

Figure 2: Saddle-node bifurcation in the system $\dot{x} = r + x^2$.[2]

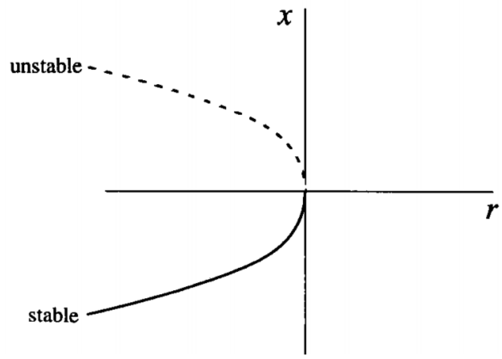

Figure 3: Bifurcation Diagram[2]

3.3 Transcritical Bifurcation

In transcritical bifurcation, as a parameter is varied, stability of a fixed point changes.
The normal form of transcritical bifurcation is - $\dot{x} = rx - x^2$. Note that $x^* = 0$ is a fixed point $\forall\ r \in \mathbb{R}$, but it is stable when $r < 0$ and unstable when $r > 0$ (*as shown in fig(4)*).

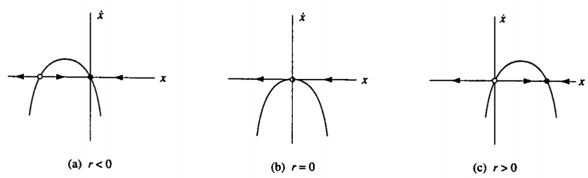

Figure 4: Transcritical bifurcation in the system $\dot{x} = rx - x^2$. [2]

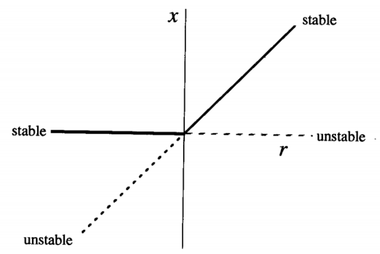

Figure 5: Bifurcation Diagram [2]

3.4 Pitchfork Bifurcation

Pitchfork bifurcations are seen in systems having spatial symmetry - in which, fixed points tend to appear and disappear in symmetrical pairs, as a parameter is varied. This will cause a transition of the system from having one fixed point to three fixed points or vice-versa.
Pitchfork bifurcation can be of two types -

3.4.1 Supercritical Pitchfork Bifurcation

Normal form : $\dot{x} = rx - x^3$
$x^* = 0$ is always a fixed point of the system, which turns from stable to unstable at $r = 0$. Also, at $r = 0$, two symmetrical stable fixed points emerge. There is a single fixed point when $r < 0$, which increases to three when $r > 0$.

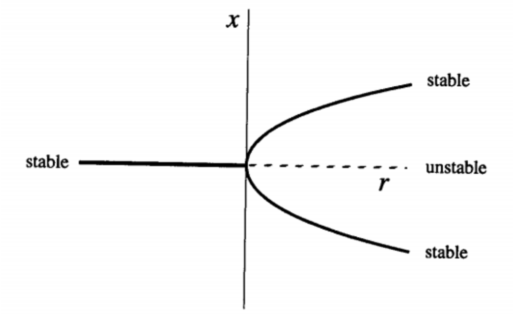

Figure 6: Bifurcation Diagram [2]

3.4.2 Subcritical Pitchfork Bifurcation

Normal form : $\dot{x} = rx + x^3$
As in the previous case, $x^* = 0$ is always a fixed point. When $r > 0$, we have a single unstable fixed point, two symmetrical unstable fixed points emerge at $r = 0$, hence giving three fixed points when $r < 0$.

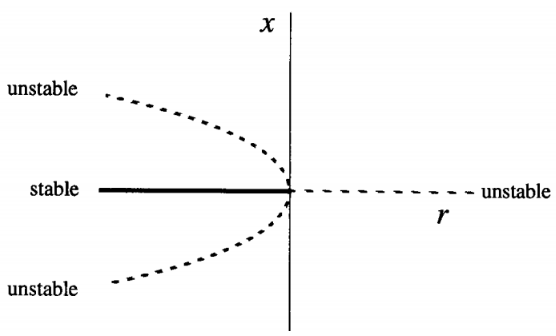

Figure 7: Bifurcation Diagram [2]

3.5 Imperfect Bifurcations and Catastrophe

Consider the system - $\dot{x} = f(x, r, h)$ - where r and h are two parameters. Suppose the system has left-right symmetry when $h = 0$ and the symmetry is broken when $h \neq 0$, then h is called the **imperfection parameter**. This causes imperfections in the bifurcation diagrams(as shown in fig(8).

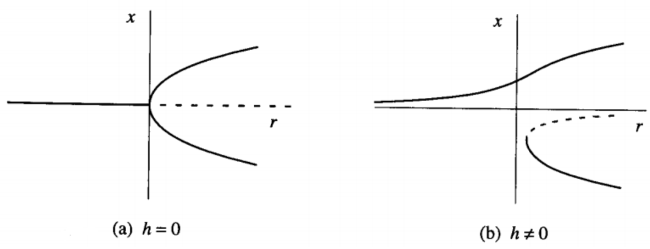

Figure 8: Bifurcation diagrams drawn in the x-r plane(for constant h) of the system $\dot{x} = h + rx - x^3$ [2]

For the system $\dot{x} = h + rx - x^3$, if we plot the fixed points x^* above the $r - h$ plane, we get the **cusp catastrophe** surface, as shown in fig(9)

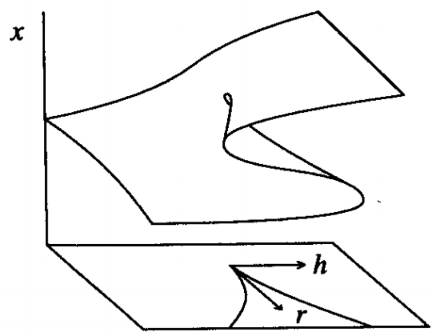

Figure 9: Cusp catastrophe surface [2]

As the parameters are varied, the state of the system is carried along the edge of the upper surface, followed by a discontinuous drop to the lower surface. This motivates the use of word *catastrophe*, as this could be catastrophic for the equilibrium of a bridge or building or an insect population.

4 Linear Systems

We start our analysis of higher order systems by analysing two-dimensional linear systems. The general form of a two-dimensional linear system is given below-

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy$$

where a,b,c and d are parameters.

Define X = $\begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then -

$$\dot{X} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\implies \dot{X} = AX \tag{3}$$

Equation 3 gives the general form of a 2-dimensional linear system in a compact way.

Remark 1. Let $X_1(t)$ and $X_2(t)$ be any two solutions of $\dot{X} = AX$, then any linear combination of X_1 and X_2 is also a solution. This is why the system is called linear.

Remark 2. $X^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a fixed point of any 2-dimensional linear system, since $X = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \dot{X} = AX = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, for any A.

4.1 Geometric Approach and Fixed Points

A two-dimensional linear system represents a vector field on a plane - at each point (x, y), a vector $(\dot{x}, \dot{y})$ is assigned. To find the trajectory starting at (x_0, y_0), we place a phase point at (x_0, y_0) and see how it is carried along the flow, as given by the vector field.

Closed orbits : If a phase point returns back to its initial point, then its trajectory is said to be a closed orbit. Hence, closed orbits represent periodic solutions of the system.

A **fixed point** is the point where $(\dot{x}, \dot{y}) = (0, 0)$. Based on their stability, there are different types of fixed points -

- **Attracting fixed point** : A fixed point x^* is said to be attracting if $\exists\, \delta > 0$ such that

  $$|x(0) - x^*| < \delta \implies \lim_{t\to\infty} x(t) = x^*.$$

- **Globally attracting fixed point** : A fixed point x^* is said to be globally attracting if

  $$\lim_{t\to\infty} x(t) = x^* \quad ,\forall x(0) \in \mathbb{R}$$

- **Liapunov stable fixed points** : A fixed point x^* is said to be Liapunov stable if $\forall\, \epsilon > 0$, $\exists\, \delta > 0$ such that -

  $$|x(0) - x^*| < \delta \implies |x(t) - x^*| < \epsilon \quad ,\forall t \geq 0.$$

- **Neutrally stable fixed points** : This type includes the fixed points which are Liapunov stable but not attracting.

- **Stable or Asymptotically stable fixed points** : This type includes the fixed points which are both Liapunov stable and attracting.

- **Unstable fixed points** : This type includes the fixed points which are neither Liapunov stable nor attracting.

4.2 Classification of Fixed Points

Lemma. Let V be an eigenvector of the matrix A with eigenvalue λ, then $X(t) = e^{\lambda t} V$ is a solution to the system $\dot{X} = AX$.

Proof. $X'(t) = e^{\lambda t}(\lambda V) = e^{\lambda t}(AV) = A(e^{\lambda t} V)$
$\implies X'(t) = AX(t)$ □

Hence, the solutions of any system $X' = AX$ will depend upon the eigenvalues and eigenvectors of A. Hence, there can be several cases -

4.2.1 Real and distinct eigenvalues

Theorem. Consider the two-dimensional linear system $X' = AX$. Let V_1 and V_2 be eigenvectors of A with eigenvalues λ_1 and λ_2 respectively such that $\lambda_1 \neq \lambda_2$. Then the general solution of the system is -

$$X(t) = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2 \quad , c_1, c_2 \in \mathbb{R} \tag{4}$$

Depending upon the sign of the two eigenvalues, fixed points can be of different types -

- **Stable node** : $\lambda_1 < 0$ and $\lambda_2 < 0$.
- **Saddle** : $\lambda_1 < 0 < \lambda_2$.
- **Unstable node** : $\lambda_1 > 0$ and $\lambda_2 > 0$.

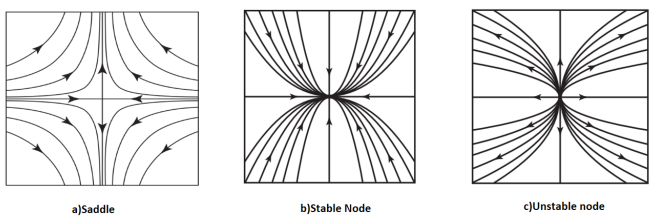

Figure 10: Phase portraits for saddle, stable and unstable nodes. [1]

4.2.2 Complex eigenvalues

Let the complex eigenvalues of A be $\alpha \pm i\beta$. Depending upon sign of α, there can be different types of fixed points :-

- **Stable spiral** : $\alpha < 0$
- **Center** : $\alpha = 0$
- **Unstable spiral** : $\alpha > 0$

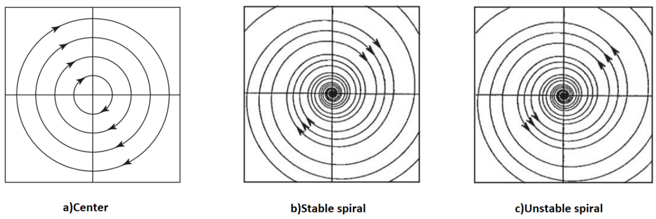

Figure 11: Phase portraits for center and spirals. [1]

4.2.3 Equal eigenvalues

- *Case 1* ($X' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} X$) : $X(t) = X_0$ is a solution to the system, $\forall\, X_0 \in \mathbb{R}^2$. Hence the phase portrait will be a whole plane of fixed points.
- *Case 2* (Non-zero equal eigenvalues with more than one linearly independent eigenvector) : In this case, the fixed point is called a **star node**.
- *Case 2* (Non-zero equal eigenvalues with only one linearly independent eigenvector) : In this case, the fixed point is called a **degenerate node**.

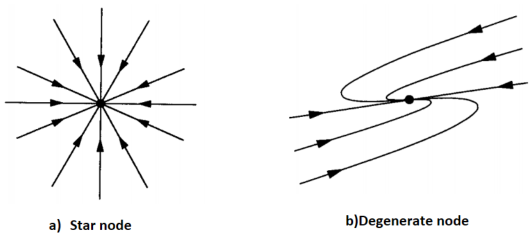

Figure 12: Phase portraits for star and degenerate nodes. [2]

4.3 Trace-Determinant Plane

Let τ and Δ be the trace and determinant of matrix A. Then the eigenvalues of A are given by -

$$\lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2}\ ;\ \lambda_2 = \frac{\tau - \sqrt{\tau^2 - 4\Delta}}{2}$$
$$\implies \tau = \lambda_1 + \lambda_2\ ;\ \Delta = \lambda_1 \lambda_2$$

- *Case 1* ($\tau^2 - 4\Delta > 0$) : λ_1 and λ_2 are real and distinct.
  - Saddle : $\Delta < 0$
  - Stable node : $\Delta > 0$ and $\tau < 0$
  - Unstable node : $\Delta > 0$ and $\tau > 0$
- *Case 2* ($\tau^2 - 4\Delta = 0$) : $\lambda_1 = \lambda_2$
  - Star node : $\lambda_1 = \lambda_2 = 0$
  - Degenerate node : $\lambda_1 = \lambda_2 \neq 0$
- *Case 3* ($\tau^2 - 4\Delta < 0$) : λ_1 and λ_2 are complex.
  - Center : $\tau = 0$
  - Stable spiral : $\tau < 0$
  - Unstable spiral : $\tau > 0$

All of the above information can be summarised in the following diagram[13] in the τ-Δ plane.

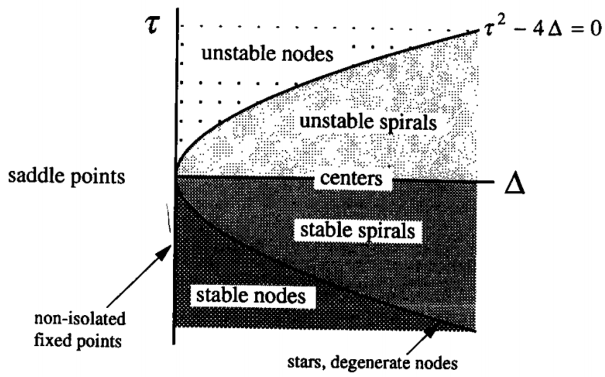

Figure 13: The τ-Δ plane [2]

5 Numerical Methods

In this section, we begin our study of non-linear dynamical systems by looking at a few numerical methods that we employ while solving such systems.
We will look at different methods for the numerical integration of the first-order system $\dot{x} = f(x)$, and will extend these methods to higher dimensions, by the end of this section. The problem that we attempt to solve is that - given the differential equation $\dot{x} = f(x)$ and the initial condition $x(t_0) = x_0$, we need a method to approximate x(t).

5.1 Euler's Method

Consider the differential equation -

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$$

In this method, we make the approximation -

$$\frac{\mathrm{d}x}{\mathrm{d}t} \approx \frac{x(t+\Delta t) - x(t)}{\Delta t}$$

$$\implies f(x_0) = \frac{x(t_0+\Delta t) - x_0}{\Delta t}$$

$$\implies x_1 = x(t_0+\Delta t) = x_0 + f(x_0)\Delta t$$

We choose Δt to be sufficiently small so that our approximation holds true and we iterate till the final time using the update rule -

$$x_{n+1} = x_n + f(x_n)\Delta t \tag{5}$$

The Euler's method is **first-order**, since the error $E = |x(t_n) - x_n|$ is proportional to the stepsize Δt ($E\ \alpha\ \Delta t$).
A python script that works based on the above algorithm is attached here - https://github.com/Ashlin-V-Thomas/Dynamical-systems/blob/main/Numerical_integration(Euler's%20method).py.

5.2 Improved Euler's Method

In order to improve the accuracy of Euler's method, we modify the update rule of iteration by finding the derivative on both ends of the time interval and using their average. For this, we find a trial value $\tilde{x}_{n+1}$, given by -

$$\tilde{x}_{n+1} = x_n + f(x_n)\Delta t$$

and the real value, x_{n+1} is found out using $\tilde{x}_{n+1}$ using -

$$x_{n+1} = x_n + \frac{1}{2}[f(x_n) + f(\tilde{x}_{n+1})]\Delta t \tag{6}$$

This method is a second-order method since ($E\ \alpha\ (\Delta t)^2$).
A python script that works based on the above algorithm is attached here - https://github.com/Ashlin-V-Thomas/Dynamical-systems/blob/main/Numerical_integration(Improved%20Euler%20method).py.

5.3 Runge - Kutta Method

It is a fourth-order method, in which we find four values -

$$k_1 = f(x_n)\Delta t$$

$$k_2 = f(x_n + \frac{1}{2}k_1)\Delta t$$

$$k_3 = f(x_n + \frac{1}{2}k_2)\Delta t$$

$$k_4 = f(x_n + k_3)\Delta t$$

Now, x_{n+1} is found out using the four values -

$$x_{n+1} = x_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{7}$$

A python script that works based on the above algorithm which solves graphs the solutions is attached here - https://github.com/Ashlin-V-Thomas/Dynamical-systems/blob/main/Numerical_integration(Runge-Kutta%20method).py.

The above method can be extended to a two-dimensional system $\dot{X} = F(X)$. The same set of computations used above for solving first-order systems can be used here, except for the fact that in this case - $x_n, k_1, k_2, k_3, k_4 \in \mathbb{R}^2$. A python script that works based on the above algorithm to solve two-dimensional system and graph the trajectory is attached here - https://github.com/Ashlin-V-Thomas/Dynamical-systems/blob/main/2-D%20Numerical%20integration.py.

6 Phase Plane Analysis

As we have seen in section 4.1, a second-order system represents a vector field on the phase plane, where we associate the vector given by $(\dot{x}, \dot{y})$ to each point (x, y). Hence by flowing along the vector field, a phase point traces out a trajectory since, the vector field gives the tangent vector of the trajectory at every point. We will use this idea to trace the trajectories of non-linear systems in the phase plane.
To begin with, let us look at a few definitions -

- **Nullclines** : Nullclines are defined as curves where $\dot{x} = 0$ (flow is vertical) or $\dot{y} = 0$ (flow is horizontal). Plotting the nullclines in the phase plane will be helpful in our analysis, as they partition the phase plane into different regions where $\dot{x}$ and $\dot{y}$ have various signs.
- **Homoclinic orbits** : Homoclinic orbits start and end at the same fixed point. That is -

$$\lim_{t \to -\infty} x(t) = x^* = \lim_{t\to\infty} x(t)$$

6.1 Existence and Uniqueness Theorem

Theorem. Consider the initial value problem $\dot{X} = F(X)$, $X(0) = X_0$, where $X \in \mathbb{R}^n$. Suppose that F is continuous and all its partial derivatives $\frac{\partial F_i}{\partial x_j}$, where $1 \leq i \leq n$ and $1 \leq j \leq n$, are continuous $\forall\ x \in D$, where $D \subset \mathbb{R}^n$ is an open connected set. Then, $\forall\ x_0 \in D$, the initial value problem has a solution x(t) for $t \in (-\tau, \tau)$ and the solution is unique.

Corollary. Consider the initial value problem $\dot{X} = F(X)$, $X(0) = X_0$, where $X \in \mathbb{R}^n$. Suppose that F is continuous and all its partial derivatives $\frac{\partial F_i}{\partial x_j}$, where $1 \leq i \leq n$ and $1 \leq j \leq n$, are continuous $\forall\, x \in D$, where $D \subset \mathbb{R}^n$ is an open connected set. Then in D, two different trajectories never intersect.

Proof. Suppose not. Then there will be two trajectories starting from the point of intersection. This is a contradiction to the uniqueness part of theorem 6.1. Hence, two trajectories can never intersect □

6.2 Fixed points and Linearization

Consider the system -

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

Let (x^*, y^*) be a fixed point of the syatem ,i.e, $f(x^*, y^*) = 0$ and $g(x^*, y^*) = 0$.
Let $\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} x(t) - x^* \\ y(t) - y^* \end{pmatrix}$ be a small disturbance from the fixed point.

$$u(t) = x(t) - x^*$$
$$\implies \dot{u} = \dot{x}$$
$$\implies \dot{u} = f(x, y) = f(x^* + u, y^* + v)$$

Now using Taylor series expansion, we obtain -

$$\dot{u} = f(x^*, y^*) + u\left(\frac{\partial f}{\partial x}\right)_{(x^*,y^*)} + v\left(\frac{\partial f}{\partial y}\right)_{(x^*,y^*)} + O(u^2, v^2, uv)$$

Since the disturbance is small, $O(u^2, v^2, uv)$ can be neglected and $f(x^*, y^*) = 0$. Hence -

$$\dot{u} \approx u\left(\frac{\partial f}{\partial x}\right)_{(x^*,y^*)} + v\left(\frac{\partial f}{\partial y}\right)_{(x^*,y^*)}$$

Similarly,

$$\dot{v} \approx u\left(\frac{\partial g}{\partial x}\right)_{(x^*,y^*)} + v\left(\frac{\partial g}{\partial y}\right)_{(x^*,y^*)}$$

Hence the disturbance $\begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$ evolves according to -

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x^*,y^*)} \begin{pmatrix} u \\ v \end{pmatrix} \tag{8}$$

The above equation gives the linearized system around the fixed point. The matrix $A = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x^*,y^*)}$ is called the **Jacobian matrix** at the fixed point (x^*, y^*).

6.2.1 Validity of linearization

Our assumption that $O(u^2, v^2, uv)$ can be neglected holds good for saddles, nodes and spirals but not for centers, stars, degenerate nodes or non-isolated fixed points as they are altered by small non-linear terms.

6.2.2 New scheme of classification of fixed points

Based on our above discussion, we can classify fixed points, based on their stability, as follows -

- **Robust cases** : These include -
  - Repellers (Sources) : Both eigenvalues have positive real part.
  - Attractors(Sinks) : Both eigenvalues have negative real part.
  - Saddles : One eigenvalue is positive and the other is negative.
- **Marginal cases** : These include -
  - Centers : Both eigenvalues are purely imaginary.
  - Higher-order and non-isolated fixed points : at least one eigenvalue is zero.

6.2.3 Hyperbolic fixed points

Definition. Let $\dot{X} = F(X)$ be an nth-order system with a fixed point X^*. X^* is said to be hyperbolic if for all eigenvalues λ_i of the linearized system around X^*, $Re(\lambda_i) \neq 0$.

Theorem (Hartman-Grobman theorem). The local phase portrait near a hyperbolic fixed point is *topologically equivalent* to the phase portrait of the linearized system.

In the above theorem, *"topologically equivalent"* means that there exists a homeomorphism (a continuous bijective function between two topological spaces that has a continuous inverse) that maps one local phase portrait onto other such that trajectories map onto trajectories and the sense of time is preserved - that is, one phase portrait is a distorted version of the other. Therfore, hyperbolic fixed points are not affected by small non-linear terms and linearization around them holds good.

Definition (***Structural Stability***). A phase portrait is said to be structurally stable if its topology cannot be changed by an arbitrarily small perturbation to the vector field.

6.3 Conservative Systems

Definition. Consider the nth-order system $\dot{X} = F(X)$. Let $E : \mathbb{R}^n \to \mathbb{R}$ be a function such that -

1. E is continuous,
2. On every trajectory, E(X) is a constant, i.e, $\frac{\partial E}{\partial t} = 0$ and
3. For all open set $D \subset \mathbb{R}^n$, $\exists\, X_1, X_2 \in D$ such that $E(X_1) \neq E(X_2)$.

Then $E(X)$ is called a conserved quantity and the system is called conservative.

Theorem. A conservative system cannot have any attracting fixed points.

Proof. Suppose not. Let X^* be an attracting fixed point. Then all trajectories starting from a neighbourhood of X^* will end up in X^*, as $t \to \infty$. Hence, $E(X)$ will be a constant on all trajectories and will be equal to $E(X^*)$. Therefore, $E(X)$ will be a constant function in that neighbourhood, but this contradicts the fact that the system is conserved, as $E(X)$ have to be non-constant on all open sets. □

Theorem (*Nonlinear centers*). Consider a second-order conservative system $\dot{X} = F(X)$, where F is continuously differentiable. Let $E(X)$ be a conserved quantity and X^* be an isolated fixed point. If X^* is a local extremum of E, then all trajectories in a small neighbourhood of X^* are closed.

6.4 Reversible Systems

Definition. Consider the nth-order system $\dot{X} = F(X)$. Let $R : \mathbb{R}^n \to \mathbb{R}^n$ be a function satisfying $R \circ R(X) = X$. If the system is invariant under the change of variables $t \to -t$ and $X \to R(X)$, then the system is said to be reversible.

Theorem (*Nonlinear centers*). Let $\dot{X} = F(X)$ be a second-order, continuously differentiable system which is reversible and has a fixed point $X^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Suppose linearization around X^* gives rise to a center, then in a small neighbourhood of X^*, all trajectories are closed curves.

6.5 Index Theory

While linearization provides us the local information about the phase portrait near a fixed point, index theory provides global information about the phase portrait.

6.5.1 Index of a closed curve

Definition. Let $\dot{X} = F(X)$ be a second-order continuously differentiable system and C be a simple closed curve (doesn't intersect itself) on the plane, that doesn't pass through any fixed points of the system. Let ϕ be the angle made by the vector field with positive x-axis at each point C, i.e.,

$$\phi = tan^{-1}\left(\frac{\dot{y}}{\dot{x}}\right)$$

Let $[\phi]_C$ denote the net change in ϕ, as we complete one counterclockwise rotation around C. Then the index of the closed curve C with respected to the vector field F, denoted by I_C, is defined as -

$$I_C = \frac{1}{2\pi}[\phi]_C \tag{9}$$

Hence, I_C is the net number of counterclockwise rotations made by the vector field during a counterclockwise traversal along the curve C.

6.5.2 Properties of Index

1. If C can be continuously deformed into C' without passing through a fixed point, then $I_C = I_{C'}$.
2. If C doesn't enclose any fixed points, then $I_C = 0$.
3. Index of a closed curve remains invariant under the change of variables $t \to -t$.
4. If C is a closed orbit, then $I_C = 1$.

6.5.3 Index of a fixed point

Definition. Let X^* be an isolated fixed point. The index of X^* is denoted by I and is defined as the index of any closed curve that encloses X_* and no other fixed points.

Index of nodes, spirals, centers, stars and degenerate nodes is $+1$, whereas, it is -1 for a saddle point.

Theorem. If a closed curve C encloses n isolated-fixed points X_i^*, $1 \leq i \leq n$, then -

$$I_C = \sum_{k=1}^{n} I_k$$

where I_k is the index of X_k^*.

Corollary. Any closed orbit in the phase plane must enclose fixed points whose indices sum to $+1$.

7 Population Growth Model - Interacting Populations

7.1 Exponential and Logistic Growth Models

Consider a species, whose population is given by the function - $N(t)$. If the resources are unlimited in its habitat, then the population evolves according to the exponential growth model -

$$\dot{N} = rN \tag{10}$$

where, $r > 0$ is the intrinsic rate of natural increase.

The above model is quite unrealistic as no population has access to unlimited resources in its habitat. So, the limited availability of resources sets an upper limit to the population, causing the population to decrease if it grows beyond that limit - that limit is called **carrying capacity(K)** for that population in that habitat. This gives rise to the logistic population growth model given by -

$$\dot{N} = rN\left(1 - \frac{N}{K}\right) \tag{11}$$

7.2 Population Interactions

Different populations living in an ecosystem will have inter-specific interactions, which can can be beneficial(+), detrimental(-) or neutral(∘) to the interacting species. Based on the outcome of interaction, we can classify population interactions as follows -

Classification of population interactions		
Population Interaction	Species A	Species B
Mutualism	+	+
Commensalism	+	∘
Predation	+	-
Parasitism	+	-
Amensalism	-	∘
Competition	-	-

Table 1: Different population interactions and their outcomes.

A population interaction is said to be a -

- **Positive Interaction** - if it is beneficial to the species(+).
- **Negative Interaction** - if it is detrimental to the species(-).
- **Neutral Interaction** - if it is neutral to the species(∘).

7.3 Mathematical Model

The system, we will be dealing with, are two-population systems, say $N_1(t)$ and $N_2(t)$ - which interact with each other. We assume that each of the populations will follow the logistic population growth model in the absence of the other.
Therefore, when $N_2 = 0$,

$$\dot{N_1} = r_1 N_1\left(1 - \frac{N_1}{K_1}\right)$$

But when $N_2 \neq 0$,

$$\dot{N_1} = r_1 N_1\left(1 - \frac{N_1}{K_1}\right) + I_1(N_1, N_2)$$

where, $I_1(N_1, N_2)$ is the **interaction term** for N_1 - that is dependent upon the nature and degree of interaction between the two populations.

7.3.1 Neutral Interaction

When the interaction is neutral to a species, its population growth won't be affected by the interaction. Hence,

$$I_1(N_1, N_2) = 0$$

7.3.2 Negative Interaction

In case of a negative interaction ,

- The interaction term will be proportional to both N_1 and N_2 since the interaction will be more extensive if the populations are higher. Hence,

$$I_1(N_1, N_2)\ \alpha\ N_1 N_2$$

- Also the interaction term will be negative since an increase in interaction will cause negative effects on the population growth.

$$I_1(N_1, N_2) < 0 \quad , \forall N_1, N_2 \in [0, \infty).$$

Incorporating all the above information, we arrive at -

$$I_1(N_1, N_2) = -\alpha_1 N_1 N_2$$

where, $\alpha_1 > 0$ is a constant of proportionality called the interaction parameter for N_1.

7.3.3 Positive Interaction

In case of a positive interaction ,

- As in the case of a negative interaction, the interaction term will be proportional to both N_1 and N_2. Hence,

$$I_1(N_1, N_2) \; \alpha \; N_1 N_2$$

- When N_1 is small, the interaction term will cause it to grow and will be positive. But this can't be the case for all values of N_1. This is because if it stays positive for very high values of N_1, it can cause the population to grow(*depending on the value of* N_2) even when N_1 has exceeded the habitat's carrying capacity - as the interaction term can dominate over the preceding term. In order to account for that, the interaction term will be controlled by the habitat's carrying capacity(K_1).

Hence,

$$I_1(N_1, N_2) = \alpha_1 N_1 N_2 \left(1 - \frac{N_1}{K_1}\right)$$

where, $\alpha_1 > 0$ is a constant of proportionality called the interaction parameter for N_1.

Before delving into the detailed analysis of each type of interaction, let us state and prove the following proposition -

Proposition 1. Consider the second order system - $\dot{X} = AX$, where A is a real matrix of the form $\begin{pmatrix} a & 0 \\ d & b \end{pmatrix}$ or $\begin{pmatrix} a & d \\ 0 & b \end{pmatrix}$, where $a \neq b$. Then, the system has -

- Saddle: if $ab < 0$
- Unstable node: if $ab > 0$ and $a + b > 0$
- Stable node: if $ab > 0$ and $a + b < 0$

Proof. For the matrix A, trace(τ) $= a + b$ and determinant(Δ) $= ab$.

$$\tau^2 - 4\Delta = (a+b)^2 - 4ab = (a-b)^2 \; > 0$$

Hence, the matrix has real and distinct eigenvalues. Therefore, the system has a saddle(if $\Delta = ab < 0$), unstable node(if $\Delta = ab > 0$ and $\tau = a + b > 0$) and stable node(if $\Delta = ab > 0$ and $\tau = a + b < 0$). □

Now with the mathematical model, let us see how populations evolve under different types of population interactions.

7.4 Mutualism

In mutualism, the interaction is positive for both the populations, hence, the system will be -

$$\dot{N_1} = (r_1 + \alpha_1 N_2) N_1 \left(1 - \frac{N_1}{K_1}\right)$$
$$\dot{N_2} = (r_2 + \alpha_2 N_1) N_2 \left(1 - \frac{N_2}{K_2}\right)$$

We can see that the system has four fixed points - $(0, 0)$, $(K_1, 0)$, $(0, K_2)$ and (K_1, K_2).
The Jacobian matrix of the system is given by -

$$\begin{pmatrix} (r_1 + \alpha_1 N_2)\left(1 - \frac{2N_1}{K_1}\right) & \alpha_1 N_1 \left(1 - \frac{N_1}{K_1}\right) \\ \alpha_2 N_2 \left(1 - \frac{N_2}{K_2}\right) & (r_2 + \alpha_2 N_1)\left(1 - \frac{2N_2}{K_2}\right) \end{pmatrix}$$

Let us look at the nature of fixed points -

Nature of fixed points		
Fixed points	Linearized system around fixed point	Nature of fixed point
$(0, 0)$	$\begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$	Unstable node
$(K_1, 0)$	$\begin{pmatrix} -r_1 & 0 \\ 0 & r_2 + \alpha_2 K_1 \end{pmatrix}$	Saddle
$(0, K_2)$	$\begin{pmatrix} r_1 + \alpha_1 K_2 & 0 \\ 0 & -r_2 \end{pmatrix}$	Saddle
(K_1, K_2)	$\begin{pmatrix} -(r_1+\alpha_1 K_2) & 0 \\ 0 & -(r_2+\alpha_2 K_1) \end{pmatrix}$	Stable node

The phase portrait for the system is shown in fig(14).

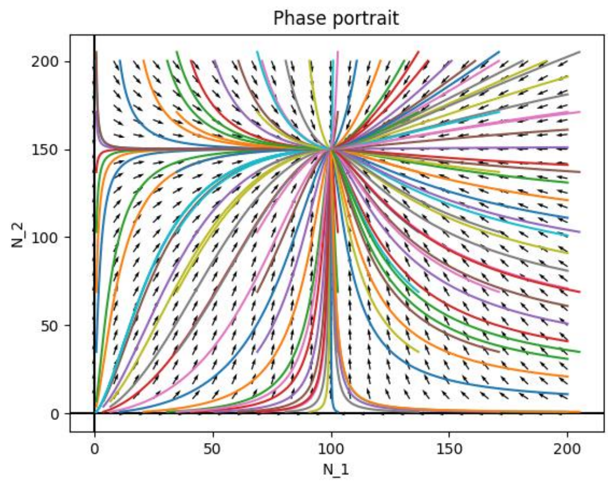

Figure 14: Phase portrait for population system with mutualism interaction ($r_1 = 2, r_2 = 3, a_1 = 0.1, a_2 = 0.18, K_1 = 100, K_2 = 150$).

Note that any trajectory that starts from any point, that doesn't lie on the axes reaches the point (K_1, K_2), since it is a stable node.
Any trajectory starting on N_1-axis - i.e., in the absence of N_2 -, ultimately reaches $(K_1, 0)$ and any trajectory on the N_2-axis - i.e., in the absence of N_1 - reaches $(0, K_2)$.
Origin (Both the populations are zero) is itself a fixed point.

7.5 Commensalism

In commensalism, the interaction is positive for one of the populations and neutral for the other. Hence, the system will be -

$$\dot{N_1} = (r_1 + \alpha_1 N_2) N_1 \left(1 - \frac{N_1}{K_1}\right)$$
$$\dot{N_2} = r_2 N_2 \left(1 - \frac{N_2}{K_2}\right)$$

We see that, the growth of the first population is affected by the second population while, the second population follows the logistic growth model.
As in the above case, the system has four fixed points - $(0, 0)$, $(K_1, 0)$, $(0, K_2)$ and (K_1, K_2).
The Jacobian matrix of the system is given by -

$$\begin{pmatrix} (r_1 + \alpha_1 N_2)\left(1 - \frac{2N_1}{K_1}\right) & \alpha_1 N_1 \left(1 - \frac{N_1}{K_1}\right) \\ 0 & r_2 \left(1 - \frac{2N_2}{K_2}\right) \end{pmatrix}$$

Let us look at the nature of different fixed points of the system -

Nature of fixed points		
Fixed points	Linearized system around fixed point	Nature of fixed point
$(0, 0)$	$\begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$	Unstable node
$(K_1, 0)$	$\begin{pmatrix} -r_1 & 0 \\ 0 & r_2 + \alpha_2 K_1 \end{pmatrix}$	Saddle
$(0, K_2)$	$\begin{pmatrix} r_1 + \alpha_1 K_2 & 0 \\ 0 & -r_2 \end{pmatrix}$	Saddle
(K_1, K_2)	$\begin{pmatrix} -(r_1 + \alpha_1 K_2) & 0 \\ 0 & -r_2 \end{pmatrix}$	Stable node

We see that both the cases of mutualism and commensalism share the same fixed points, with the same stability, hence, their phase portraits are also similar.

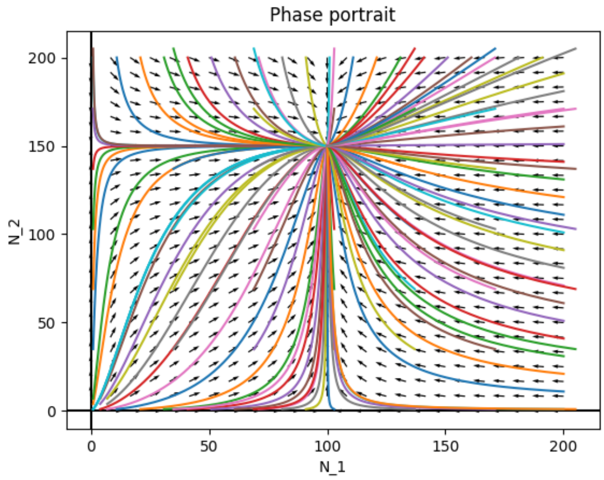

Figure 15: Phase portrait for population system with commensalism interaction($r_1 = 2, r_2 = 3, a_1 = 0.1, a_2 = 0.18, K_1 = 100, K_2 = 150$).

Even though the phase portraits look similar, one should admire the fact that - in commensalism, the growth of second population is not affected by the first but in mutualism, the growth of second population is accelerated by an increase in the first population. But in both cases, the populations eventually reach their carrying capacities.

7.6 Amensalism

In amensalism, the interaction is negative for one of the populations and neutral for the other. Hence, the system will be -

$$\dot{N_1} = r_1 N_1 \left(1 - \frac{N_1}{K_1}\right) - \alpha_1 N_1 N_2$$
$$\dot{N_2} = r_2 N_2 \left(1 - \frac{N_2}{K_2}\right)$$

On solving the simultaneous equations - $\dot{N_1} = 0$ and $\dot{N_2} = 0$ - we find that the system has -

- 3 fixed points (when $r_1 \leq \alpha_1 K_2$) - $(0, 0)$, $(K_1, 0)$ and $(0, K_2)$.
- 4 fixed points (when $r_1 > \alpha_1 K_2$) - $(0, 0)$, $(K_1, 0)$, $(0, K_2)$ and $\left(\frac{(r_1 - \alpha_1 K_2) K_1}{r_1}, K_2\right)$.

The Jacobian matrix of the system is -

$$\begin{pmatrix} r_1 \left(1 - \frac{2N_1}{K_1}\right) - \alpha_1 N_2 & -\alpha_1 N_1 \\ 0 & r_2 \left(1 - \frac{2N_2}{K_2}\right) \end{pmatrix}$$

At $(0,0)$, the linearized system is $\begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$ and hence $(0,0)$ is always an unstable node.
Similarly, the linearized system around $(K_1, 0)$ is $\begin{pmatrix} -r_1 & -\alpha_1 K_1 \\ 0 & r_2 \end{pmatrix}$. By proposition 1, the fixed point is a saddle, since $-r_1 r_2 < 0$.

Case 1 : $r_1 < \alpha_1 K_2$

Linearized system around $(0, K_2)$ is $\begin{pmatrix} r_1 - \alpha_1 K_2 & 0 \\ 0 & -r_2 \end{pmatrix}$ and since $r_1 - \alpha_1 K_2 < 0$ and $-r_2 < 0$, $(0, K_2)$ is a stable node.

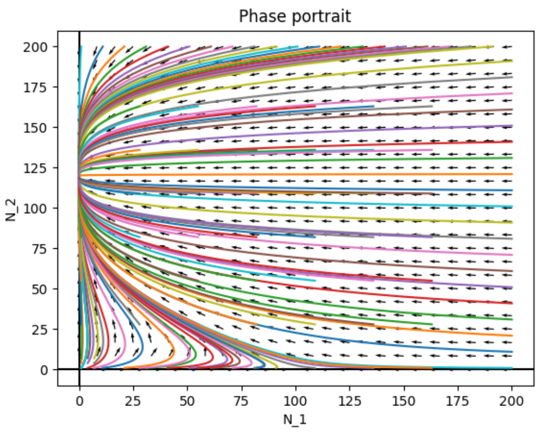

Figure 16: Phase portrait for population system with amensalism interaction with $r_1 < \alpha_1 K_2$ ($r_1 = 2, r_2 = 3, a_1 = 0.1, a_2 = 0.18, K_1 = 100, K_2 = 120$).

We see that most of the trajectories converge to the stable node $(0, 120)$,i.e, N_1 goes extinct and N_2 attains carrying capacity.

Case 2 : $r_1 = \alpha_1 K_2$

The linearized system around $(0, K_2)$ is $\begin{pmatrix} 0 & 0 \\ 0 & -r_2 \end{pmatrix}$ (*The fixed point is not hyperbolic and linearization may not be valid*). That is, $\dot{N_1} = 0$ and $\dot{N_2} = -r_2 N_2$. On solving the above system, we get -

$$N_1(t) = N_1(0) \quad \forall t \geq 0$$
$$N_2(t) = N_2(0) e^{-r_2 t}$$

We compute the phase portrait numerically - shown in fig(17).

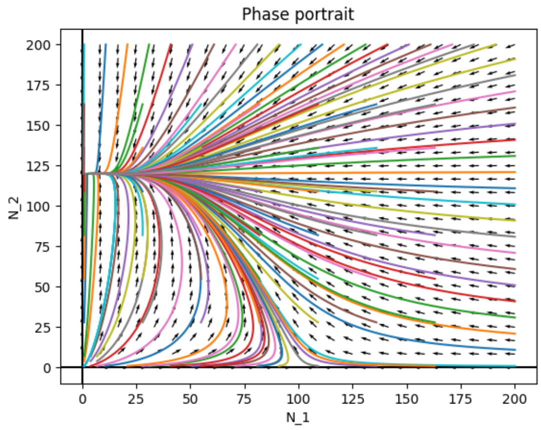

Figure 17: Phase portrait for population system with amensalism interaction with $r_1 = \alpha_1 K_2$ ($r_1 = 2, r_2 = 3, a_1 = \frac{1}{60}, a_2 = 0.18, K_1 = 100, K_2 = 120$).

Hence we see that the nature of phase portrait around $(0, K_2)$ is similar to that of the linearized system.

Case 3 : $r_1 > \alpha_1 K_2$

Linearized system around $(0, K_2)$ is $\begin{pmatrix} r_1 - \alpha_1 K_2 & 0 \\ 0 & -r_2 \end{pmatrix}$ and since $r_1 - \alpha_1 K_2 > 0$ and $-r_2 < 0$, $(0, K_2)$ is a saddle.

In this case, there is a fourth fixed point - $\left(\frac{(r_1 - \alpha_1 K_2) K_1}{r_1}, K_2\right)$ and the linearized system around it is $\begin{pmatrix} -r_1 + \alpha_1 K_2 & -\frac{\alpha_1 K_1}{r_1}(r_1 - \alpha_1 K_2) \\ 0 & -r_2 \end{pmatrix}$.
By proposition 1, the fixed point is a stable node, since $r_2(r_1 - \alpha_1 K_2) > 0$ and $\alpha_1 K - 2 - r_1 - r_2 < 0$.

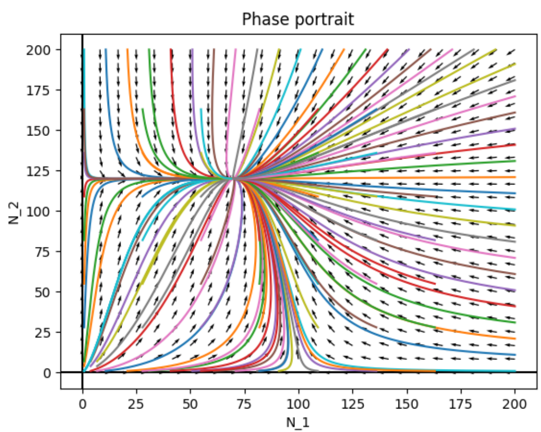

Figure 18: Phase portrait for population system with amensalism interaction with $r_1 = \alpha_1 K_2$ ($r_1 = 2, r_2 = 3, a_1 = 0.005, a_2 = 0.18, K_1 = 100, K_2 = 120$).

7.7 Predation / Parasitism

In both predation and parasitism, the interaction is positive for one of the populations(predator/parasite) and negative for the other(prey/host). Hence, the mathematical model for both the interactions would be the same but depending upon the nature of interaction, the parameters will decide how the system evolves and hence, the parameters will draw margins between predation and parasitism.
The system will be -

$$\dot{N_1} = (r_1 + \alpha_1 N_2) N_1 \left(1 - \frac{N_1}{K_1}\right)$$
$$\dot{N_2} = r_2 N_2 \left(1 - \frac{N_2}{K_2}\right) - \alpha_2 N_1 N_2$$

On solving the simultaneous equations - $\dot{N_1} = 0$ and $\dot{N_2} = 0$ - we find that the system has -

- 3 fixed points (when $r_2 \leq \alpha_2 K_1$) - $(0,0)$, $(K_1, 0)$ and $(0, K_2)$.
- 4 fixed points (when $r_2 > \alpha_2 K_1$) - $(0,0)$, $(K_1, 0)$, $(0, K_2)$ and $\left(K_1, \frac{K_2(r_2 - \alpha_2 K_1)}{r_2}\right)$.

The Jacobian matrix of the system is -

$$\begin{pmatrix} (r_1 + \alpha_1 N_2)\left(1 - \frac{2N_1}{K_1}\right) & \alpha_1 N_1 \left(1 - \frac{N_1}{K_1}\right) \\ -\alpha_2 N_2 & r_2\left(1 - \frac{2N_2}{K_2}\right) - \alpha_2 N_1 \end{pmatrix}$$

At $(0,0)$, the linearized system is $\begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$ and hence $(0,0)$ is always an unstable node.

Similarly, the linearized system around $(0, K_2)$ is $\begin{pmatrix} r_1 + \alpha_1 K_1 & 0 \\ -\alpha_2 K_2 & -r_2 \end{pmatrix}$. By proposition 1, the fixed point is a saddle, since $-r_2(r_1 + \alpha_1 K_1) < 0$.

Case 1 : $r_2 < \alpha_2 K_1$

Linearized system around $(K_1, 0)$ is $\begin{pmatrix} -r_1 & 0 \\ 0 & r_2 - \alpha_2 K_1 \end{pmatrix}$ and since $-r_1 < 0$ and $r_2 - \alpha_2 K_1 < 0$, $(K_1, 0)$ is a stable node.

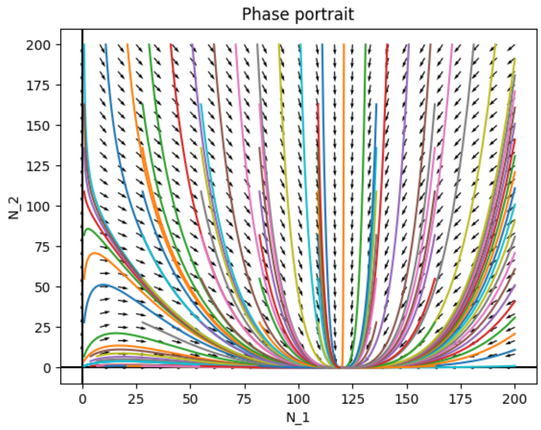

Figure 19: Phase portrait for population system with predation or parasitism interaction with $r_2 < \alpha_2 K_1$ ($r_1 = 3, r_2 = 2, a_1 = 0.18, a_2 = 0.1, K_1 = 120, K_2 = 100$).

Case 2 : $r_2 = \alpha_2 K_1$

The linearized system around $(K_1, 0)$ is $\begin{pmatrix} -r_1 & 0 \\ 0 & 0 \end{pmatrix}$ (*The fixed point is not hyperbolic and linearization may not be valid*).
On solving the above system, we get -

$$N_1(t) = N_1(0) e^{-r_1 t}$$
$$N_2(t) = N_2(0) \quad \forall t \geq 0$$

We compute the phase portrait numerically - shown in fig(17).

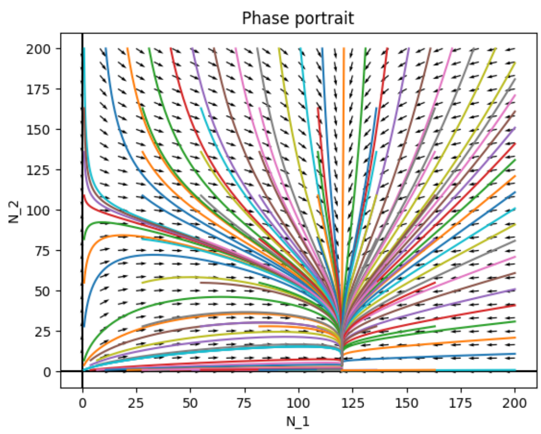

Figure 20: Phase portrait for population system with predation or parasitism interaction with $r_2 = \alpha_2 K_1$ ($r_1 = 3, r_2 = 2, a_1 = 0.18, a_2 = \frac{1}{60}, K_1 = 120, K_2 = 100$).

Hence we see that the nature of phase portrait around $(K_1, 0)$ is similar to that of the linearized system.

Case 3 : $r_2 > \alpha_2 K_1$

Linearized system around $(K_1, 0)$ is $\begin{pmatrix} -r_1 & 0 \\ 0 & r_2 - \alpha_2 K_1 \end{pmatrix}$ and since $r_2 - \alpha_2 K_1 > 0$ and $-r_1 < 0$, $(K_1, 0)$ is a saddle.

In this case, there is a fourth fixed point - $\left(K_1, \frac{K_2(r_2 - \alpha_2 K_1)}{r_2}\right)$ and the linearized system around it is -

$$\begin{pmatrix} -\left(r_1 + \frac{\alpha_1 K_2}{r_2}(r_2 - \alpha_2 K_1)\right) & 0 \\ -\frac{\alpha_2 K_2}{r_2}(r_2 - \alpha_2 K_1) & \alpha_2 K_1 - r_2 \end{pmatrix}$$

.

By proposition 1, the fixed point is a stable node, since the determinant of the linearized system is positive and its trace is negative.

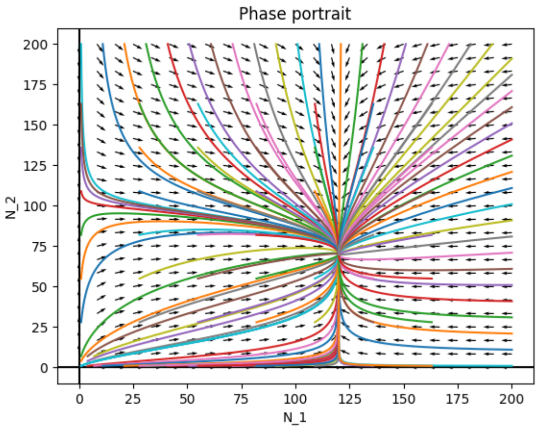

Figure 21: Phase portrait for population system with predation or parasitism interaction with $r_2 < \alpha_2 K_1$ ($r_1 = 3, r_2 = 2, a_1 = 0.18, a_2 = 0.005, K_1 = 120, K_2 = 100$).

7.8 Competition

In competition, the interaction is negative for both the populations, hence, the system will be -

$$\dot{N}_1 = r_1 N_1 \left(1 - \frac{N_1}{K_1}\right) - \alpha_1 N_1 N_2$$
$$\dot{N}_2 = r_2 N_2 \left(1 - \frac{N_2}{K_2}\right) - \alpha_2 N_1 N_2$$

On solving the simultaneous equations - $\dot{N}_1 = 0$ and $\dot{N}_2 = 0$ - we find that the system has -

- 4 fixed points (when $r_2 < \alpha_2 K_1$ and $r_1 < \alpha_1 K_2$ or $r_2 > \alpha_2 K_1$ and $r_1 > \alpha_1 K_2$) - $(0, 0)$, $(K_1, 0)$, $(0, K_2)$ and $\left(\frac{r_2 K_1(\alpha_1 K_2 - r_1)}{\alpha_1 \alpha_2 K_1 K_2 - r_1, r_2}, \frac{r_1 K_2(\alpha_2 K_1 - r_2)}{\alpha_1 \alpha_2 K_1 K_2 - r_1, r_2}\right)$.
- 3 fixed points (otherwise) - $(0, 0)$, $(K_1, 0)$ and $(0, K_2)$.

The Jacobian matrix of the system is -

$$\begin{pmatrix} r_1\left(1 - \frac{2N_1}{K_1}\right) - \alpha_1 N_2 & -\alpha_1 N_1 \\ -\alpha_2 N_2 & r_2\left(1 - \frac{2N_2}{K_2}\right) - \alpha_2 N_1 \end{pmatrix}$$

At $(0, 0)$, the linearized system is $\begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$ and hence $(0, 0)$ is always an unstable node.

Linearized system around $(K_1, 0)$ is $\begin{pmatrix} -r_1 & -\alpha_1 K_1 \\ 0 & r_2 - \alpha_2 K_1 \end{pmatrix}$. We have analysed a similar system in 7.7 using proposition 1 and had concluded that - $(K_1, 0)$ is a saddle if $r_2 - \alpha_2 K_1 > 0$ and stable node if $r_2 - \alpha_2 K_1 < 0$ and results in an exponentially decaying N_1 and constant N_2(around $(K_1, 0)$) if $r_2 - \alpha_2 K_1 = 0$.

Linearized system around $(0, K_2)$ is $\begin{pmatrix} r_1 - \alpha_1 K_2 & 0 \\ -\alpha_2 K_2 & -r_2 \end{pmatrix}$. We have analysed a similar system in 7.6 using proposition 1 and had concluded that - $(0, K_2)$ is a saddle if $r_1 - \alpha_1 K_2 > 0$ and stable node if $r_1 - \alpha_1 K_2 < 0$ and results in an exponentially decaying N_2 and constant N_1(around $(0, K_2)$) if $r_1 - \alpha_1 K_2 = 0$.

If $r_2 < \alpha_2 K_1$ and $r_1 < \alpha_1 K_2$ or $r_2 > \alpha_2 K_1$, $r_1 > \alpha_1 K_2$, we have a fourth fixed point (N_1^*, N_2^*), where $N_1^* = \frac{r_2 K_1(\alpha_1 K_2 - r_1)}{\alpha_1 \alpha_2 K_1 K_2 - r_1, r_2}$ and $N_2^* = \frac{r_1 K_2(\alpha_2 K_1 - r_2)}{\alpha_1 \alpha_2 K_1 K_2 - r_1, r_2}$. Linearization around it will give us the system -

$$A = \begin{pmatrix} -\frac{r_1 N_1^*}{K_1} & -\alpha_1 N_1^* \\ -\alpha_2 N_2^* & -\frac{r_2 N_2^*}{K_2} \end{pmatrix}$$

Let $\Delta = det(A)$ and $\tau = trace(A)$.

$$\Delta = \frac{r_1 N_1^* r_2 N_2^*}{K_1 K_2} - \alpha_1 \alpha_2 N_1^* N_2*$$
$$\Delta = \frac{N_1^* N_2*}{K_1 K_2}(r_1 r_2 - \alpha_1 \alpha_2 K_1 K_2)$$

Case 1: $r_2 < \alpha_2 K_1$ **and** $r_1 < \alpha_1 K_2$

In this case, $(r_1 r_2 - \alpha_1 \alpha_2 K_1 K_2) < 0$ and hence $\Delta < 0$.

$$\implies \tau^2 - 4\Delta > 0$$

So, the system has real and distinct eigenvalues and $\Delta < 0$ and hence the fixed point is a saddle.

Case 2: $r_2 > \alpha_2 K_1$ **and** $r_1 > \alpha_1 K_2$

In this case, $(r_1 r_2 - \alpha_1\alpha_2 K_1 K_2) > 0$ and hence $\Delta > 0$.

$$\tau^2 - 4\Delta = \left(\frac{r_1 N_1}{K_1} + \frac{r_2 N_2}{K_2}\right)^2 - \frac{4 r_1 r_2 N_1^* N_2^*}{K_1 K_2} + 4\alpha_1\alpha_2 N_1^* N_2^*$$

$$\implies \tau^2 - 4\Delta = \left(\frac{r_1 N_1}{K_1} - \frac{r_2 N_2}{K_2}\right)^2 + 4\alpha_1\alpha_2 N_1^* N_2* > 0$$

Hence, the system has real and distinct eigenvalues.

$$\tau = -\left(\frac{r_1 N_1}{K_1} + \frac{r_2 N_2}{K_2}\right) < 0$$

Since $\tau < 0$ and $\Delta > 0$, the fixed point is a stable node.

We have included a few phase portraits for competing two-population systems, each with a different set of parameters, obtained from numerical methods and we can find that those are in agreement with our above analysis.

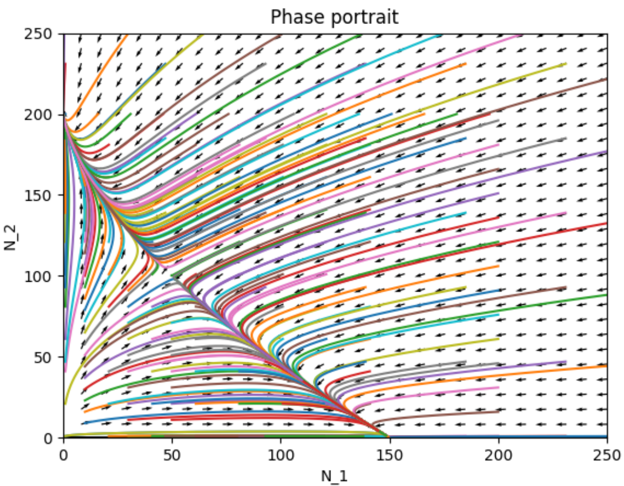

Figure 22: Phase portrait for population system with competition interaction with $(r_1 = 3, r_2 = 1, a_1 = 0.02, a_2 = 0.01, K_1 = 150, K_2 = 200)$. Here, $r_2 < \alpha_2 K_1$ and $r_1 < \alpha_1 K_2$.

In the above phase portrait, we see that either one of the two populations go extinct as time passes on and the other one reaches its carrying capacity and we have a saddle in the phase portrait. The fate of the system,i.e., which population will go extinct depends on the initial condition.

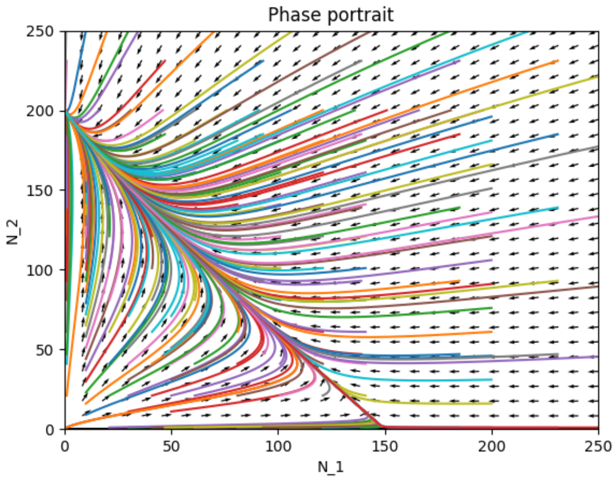

Figure 23: Phase portrait for population system with competition interaction with $(r_1 = 3, r_2 = 2, a_1 = 0.02, a_2 = 0.01, K_1 = 150, K_2 = 200)$. Here, $r_2 > \alpha_2 K_1$ and $r_1 < \alpha_1 K_2$.

The above phase portrait has a saddle at $(150, 0)$ and a stable node at $(0, 200)$. Also, we see that for most trajectories - N_1 gets extinct and N_2 reaches its carrying capacity $K_2 = 200$.

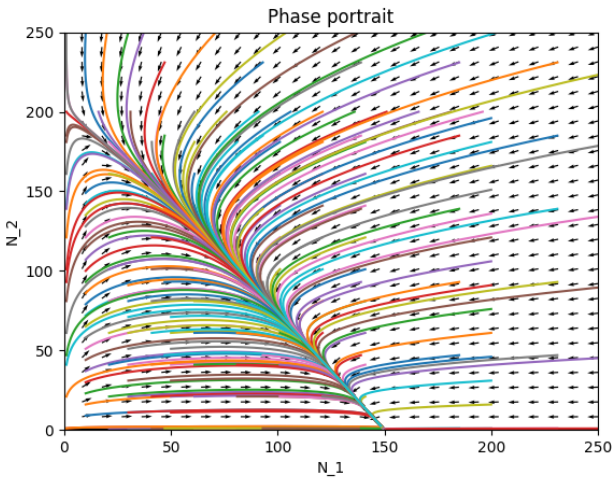

Figure 24: Phase portrait for population system with competition interaction with $(r_1 = 5, r_2 = 1, a_1 = 0.02, a_2 = 0.01, K_1 = 150, K_2 = 200)$. Here, $r_2 < \alpha_2 K_1$ and $r_1 > \alpha_1 K_2$.

The above phase portrait has a saddle at $(0, 200)$ and a stable node at $(150, 0)$. Also, we see that for most trajectories - N_2 gets extinct and N_1 reaches its carrying capacity $K_1 = 150$.

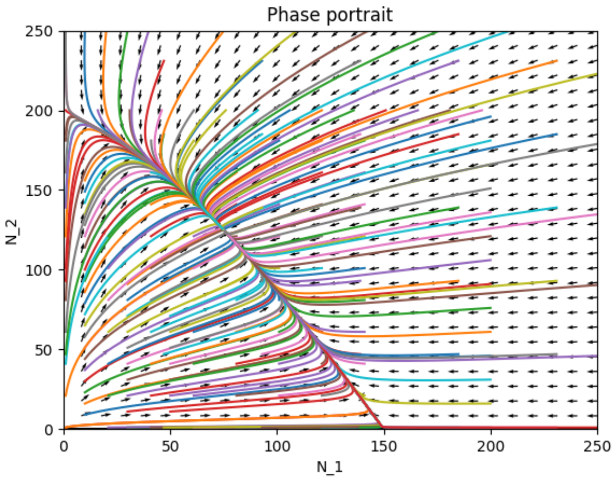

Figure 25: Phase portrait for population system with competition interaction with ($r_1 = 5, r_2 = 2, a_1 = 0.02, a_2 = 0.01, K_1 = 150, K_2 = 200$). Here, $r_2 > \alpha_2 K_1$ and $r_1 > \alpha_1 K_2$.

We see that the system has a stable node at $(75, 125)$ and saddles at $(0, 200)$ and $(150, 0)$. Therefore, most trajectories reaches the fixed point $(75, 125)$, without any of the populations going extinct.

We see that in figures 22, 23 and 24, as the system attains equilibrium, one of the populations go extinct. This is in agreement with *Gause's Competitive Exclusion Principle* - which states that two closely related species competing for the same resources cannot co-exist indefinitely and the competitively inferior one will be eliminated eventually.

But latest studies have shown that there are competing species which promote coexistence rather than exclusion and they have evolved methods for resource partitioning. We see that the model explains this phenomena as shown in fig(25) where $r_2 > \alpha_2 K_1$ and $r_1 > \alpha_1 K_2$.

We have used a python script to simulate the growth of population systems and graph it. The code is attached here - https://github.com/Ashlin-V-Thomas/Dynamical-systems/blob/main/Population%20growth%20simulator.py

8 Conclusion

In this report, we have discussed about the nonlinear systems around us, different graphical and numerical methods to analyze them and the different types of bifurcations that can happen in such systems. We have also taken a look at linear systems and have seen under what conditions, we can approximate a nonlinear system to its linearized form. Finally, we concluded by analysing interacting 2-population systems where we made use of the knowledge that we had acquired so far.

References

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[2] Steven H. Strogatz. *Nonlinear Dynamics and Chaos*. CRC press, 1994.