The "Splitwise" Problem

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1 Problem Statement

Consider a team G of n persons. Within their team, they make payments for each other and payback once all the payments are done. The problem is to find the minimum number of payments in which the repayment can be completed.

2 Reduction to a non-directed problem

Definition (Contribution) : Contribution of a person $P \in G$ is defined as the net amount[1] spent (negative) or received (positive) by the person after all the transactions are complete and is denoted by $C(P)$.
Definition (NPC) : $P \in G$ is said to be an NPC if $C(P) = 0$.

Given any splitwise problem, it can be reduced to a **non-directed problem**. Let $G' = \{P_1, P_2, ..., P_m\}$ be the non-NPCs in G, then the non-directed problem is stated as follows - what is the minimum number of payments that has to be carried out in G' so that $C(P_i)$ remains the same as that of the given splitwise problem for all $i \in [k]$?

Lemma 2.1: The solutions for a given splitwise problem and the corresponding non-directed problem are the same.

Proof : The final result of both the splitwise problem and its corresponding non-directed problem are the same since the contribution of each person remains the same in both the problems. Hence, if there is an algorithm which settles all transactions in one problem, it can be used in the other as well. Therefore, the minimum number of payments required to settle all transactions in both the problems are same, hence, their solutions are the same.

[1] including the transactions involved in repayment only.

3 Initial Approach : An algorithm

Theorem 3.1: Given any splitwise problem in a group of n persons, the transactions can be settled in $(n-1)$ payments.

Proof : Consider the corresponding non-directed problem, it will have m persons, $m \leqslant n$. Let $\mathbb{P} = \{P_1, P_2, ..., P_u\}$ be the payers and $\mathbb{R} = \{R_1, R_2, ..., R_v\}$ be the recipients , then $u + v = m$.
Choose P_1 , and R_1, execute the payment from P_1 to R_1, such that at least one of them gets settled. If $C(P_1) > C(R_1)$, R_1 gets settled and if $C(P_1) < C(R_1)$, P_1 gets settled. Finally, if $C(P_1) = C(R_1)$, both gets settled. Hence, any such payment can settle at least one among the payer and the recipient.
In the worst case scenario , $(u-1)+(v-1)$ payments can reduce the sets $\mathbb{P}$ and $\mathbb{R}$ to singleton sets. Now, this can be settled in one payment(Since, $\sum_{i=1}^{u} C(P_i) = \sum_{i=1}^{v} C(R_i)$).

$$\begin{aligned}\text{Number of payments done} &= (u-1)+(v-1)+1 \\ &= (u+v)-1 = m-1 \leqslant n-1\end{aligned}$$

Hence, any splitwise problem can be settled in $(n-1)$ payments.

4 Finding the lower bound

Notation : Given any splitwise problem, $\mathbb{P}$ will denote the set of payers and $\mathbb{R}$ will denote the set of recipients. Let $A \subset \mathbb{P}$ or $A \subset \mathbb{R}$, then $S(A)$ is defined as -

$$S(A) = \sum_{x \in A} C(x)$$

Definition (Cancelling pair): Let $A \subset \mathbb{P}$ and $B \subset \mathbb{R}$ be non-empty. (A, B) is said to be a cancelling pair if $S(A) + S(B) = 0$. A cancelling pair is proper if $A \neq \mathbb{P}$ and $B \neq \mathbb{R}$

Definition (Independent cancelling pair): Let (A_1, B_1) and (A_2, B_2) be two cancelling pairs of a splitwise problem. The two pairs are said to be independent if $A_1 \cap A_2 = B_1 \cap B_2 = \varnothing$.

Definition (Partition): Let $\mathcal{P} = \{(A_1, B_1), (A_2, B_2), ..., (A_n, B_n)\}$ be a set of cancelling pairs of a splitwise problem which are pairwise independent and $A_1 \cup A_2 .... \cup A_n = \mathbb{P}$ and $B_1 \cup B_2 .... \cup B_n = \mathbb{R}$. The $\mathcal{P}$ is called a partition of the splitwise problem.
Example : Let (A, B) be a cancelling pair, then $\{(A, B), (\mathbb{P} \backslash A, \mathbb{R} \backslash B)\}$ is a partition of the splitwise problem.

Definition (Fundamental Partition): Let $\mathcal{F}$ be the partition of A having the largest cardinality among all partitions of the problem, then $\mathcal{F}$ is called a fundamental partition of the problem.

Definition (Complete problem): A splitwise problem is said to be complete if its fundamental partition is the singleton set - $\{(\mathbb{P}, \mathbb{R})\}$.

Lemma 4.1: A splitwise problem is complete if and only if it has no proper cancelling pairs.
Proof: Suppose there exists a proper cancelling pair (A, B) for a complete problem, then $\{(A, B), (\mathbb{P}\backslash A, \mathbb{R}\backslash B)\}$ is a partition of the problem having a cardinality greater than its fundamental partition since the fundamental partition is a singleton set due to completeness. This contradicts the definition of a fundamental partition. Hence, it has no proper cancelling pair. The reverse implication follows from the definition.

Lemma 4.2: Let $(A, B) \in \mathcal{F}$, where $\mathcal{F}$ is the fundamental partition of a splitwise problem. Then, the splitwise problem with $\mathbb{P} = A$ and $\mathbb{R} = B$ is complete.
Proof: Suppose not. It follows from *lemma 4.1* that we can find a cancelling pair (A', B') of the new problem. Then, $\{(A', B'), (A\backslash A', B\backslash B')\}$ is a partition of (A, B). Therefore, $(\mathcal{F}\backslash\{(A, B)\}) \cup \{(A', B'), (A\backslash A', B\backslash B')\}$ is a partition of the original problem and the cardinality of this partition is greater than that of $\mathcal{F}$. This contradicts that $\mathcal{F}$ is the fundamental partition. Hence, the (A, B) problem is complete.

Theorem 4.1: Let n be the number of non-NPC members in a complete splitwise problem. Then the minimum number of payments in which the problem can be settled is $n - 1$.
Proof: Suppose it can be settled in less than $n - 1$ payments, then there is at least one payment, apart from the final payment, that settles both the payer and recipient. Consider one among those, let the payer and recipient of the payment be P_0 and R_0 respectively. Let $P' = \{P_1, P_2, ..., P_u\}$ be the payers who paid to R_0, prior to P_0. For minimum number of payments, at least one of the two participants should be settled in a payment *(Else, this cannot be compensated in future as we can only settle a maximum of 2 persons by one payment)*. Hence, $P_1, P_2, ..., P_u$ are settled by payment with R_0. Hence ,

$$C(R_0) + C(P_1) + C(P_2) + ... + C(P_u) + C(P_0) = 0$$

Then, $(P', \{R_0\})$ is a cancelling pair. Since the payment we are considering is not the final payment, $P' \neq \mathbb{P}$. Hence, the cancelling pair is proper. This contradicts *lemma 4.1*.
Therefore, the minimum number of payments required to settle the problem is $n - 1$.

Theorem 4.2: Consider a splitwise problem with n non-NPC members and let m be the cardinality of its fundamental partition. Then, the minimum number of payments in which the problem can be settled is $n - m$.
Proof: Let $\mathcal{F} = \{(A_1, B_1), ..., (A_m, B_m)\}$ be its fundamental partition. Let n_i

and m_i respectively denote the cardinalities of A_i and B_i $\forall i \in \{1, 2, ..., m\}$. Since $\mathcal{F}$ is a partition, we can consider settling the problem as settling m independent problems since the cancelling pairs in a partition are independent. Hence, the minimum number of ways in which the original problem can be solved is same as the minimum number of ways in which we can solve all the m independent problems *(This is because all the problems are independent and settling one doesn't affect the other. Even though one rearranges $\mathcal{F}$ to produce a new partition, this will still hold as $\mathcal{F}$ is the fundamental partition and any new such partition will have cardinality less than or equal to that of $\mathcal{F}$).* By *lemma 4.2*, all the m problems are complete and hence by *theorem 4.1*, ith problem can be settled in a minimum of $n_i + m_i - 1$ payments. Hence,

$$\text{Minimum number of payments required to settle the whole problem}$$
$$= \sum_{i=1}^{m}(n_i + m_i - 1) = \sum_{i=1}^{m}(n_i + m_i) - m = n - m \qquad \blacksquare$$